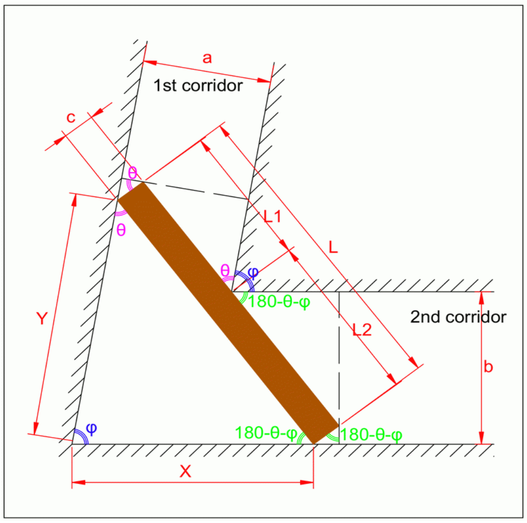
MAXIMUM LENGTH OF LADDER (WITH WIDTH) TO GO AROUND ANY CORNER



The problem of the maximum length of a ladder (with no width ie. a single line) to around a 90-degree corner in a corridor is quite popular in the web. This length is equal to $(a^{2/3}+b^{2/3})^{3/2}$

But, if the ladder (or a piece of furniture for that matter) has a certain width, the problem is more complicated and has no analytical solution, since this would require to solve a 6^{th} degree equation. This problem is more complicated if the corner is not 90^{o} but has some other value. This also requires a 6^{th} degree equation but definitely more complicated.

Since this might be a problem in practical every day applications, here we derive the appropriate equation and solve it with numerical analysis, since there exists no analytical solution for a sixth degree equation.

Lets suppose that the width of the 1^{st} corridor is *a*, the width of the 2^{nd} corridor is *b*, the angle of the corner is *φ* and the width of the ladder (or furniture) is *c*. Obviously, both *a* and *b* should be greater than *c*. At some critical stage, the ladder is wedged between the walls and the corner as seen in figure. The length should be such that if the ladder is pushed forward of backward with its long side touching the two outer walls, the other long side of it would move clear of the edge of the corner.

The length of the ladder L is the sum of the segments L1 and L2 where $L1=\frac{(a-c\cdot\cos(\vartheta))}{\sin(\vartheta)}$ and

$$L2=\frac{(b+c\cdot\cos(\phi+\vartheta))}{\sin(\phi+\vartheta)}$$

So: $L=\frac{(a-c\cdot\cos(\vartheta))}{\sin(\vartheta)}+\frac{(b+c\cdot\cos(\phi+\vartheta))}{\sin(\phi+\vartheta)}$ or, $L=\frac{a}{\sin(\theta)}-c\cdot ctn(\theta)+\frac{b}{\sin(\theta+\phi)}+c\cdot ctn(\phi+\theta)$ (1)

The first derivative of L is:

$$L'=-\frac{a\cdot\cos(\theta)}{\sin^2(\theta)}+c\cdot\frac{1}{\sin^2(\theta)}-\frac{b\cdot\cos(\theta+\phi)}{\sin^2(\theta+\phi)}-c\cdot\frac{1}{\sin^2(\theta+\phi)}$$ or

$$L'=\frac{-a\cdot\cos(\theta)\cdot\sin^2(\phi+\theta)+c\cdot\sin^2(\phi+\theta)-b\cdot\sin^2(\theta)\cdot\cos(\theta+\phi)-c\cdot\sin^2(\theta)}{\sin^2(\theta)\cdot\sin^2(\phi+\theta)} \quad (2)$$

To maximize L, the first derivative must be zero, so the nominator of (2) should be zero. After replacing: $\sin(\phi+\theta)=\sin(\phi)\cdot\cos(\theta)+\cos(\phi)\cdot\sin(\theta)$ and $\cos(\phi+\theta)=\cos(\phi)\cdot\cos(\theta)-\sin(\phi)\cdot\sin(\theta)$ the nominator of (2), as a function of θ to be solved is:

$$f'(\theta)=-a\cdot\cos(\theta)\cdot[(\cos(\phi)\cdot\sin(\theta)+\sin(\phi)\cdot\cos(\theta))]^2+c\cdot[\sin(\phi)*\cos(\theta)+\cos(\phi)\cdot\sin(\theta)]^2$$
$$-b\cdot\sin^2(\theta)\cdot[(\cos(\phi)\cdot\cos(\theta)-\sin(\phi)\cdot\sin(\theta))]-c\cdot\sin^2(\theta)$$

or

$$f'(\theta)=-a\cdot\cos^2(\phi)\cdot\cos(\theta)\cdot\sin^2(\theta)-a\cdot\sin(2\cdot\phi)\cdot\cos^2(\theta)\cdot\sin(\theta)-a\cdot\sin^2(\phi)\cdot\cos^3(\theta)+\ldots$$
$$..+c\cdot\cos^2(\phi)\cdot\sin^2(\theta)+c\cdot\sin(2\cdot\phi)\cdot\sin(\theta)\cdot\cos(\theta)+c\cdot\sin^2(\phi)\cdot\cos^2(\theta)\ldots$$
$$..-b\cdot\cos(\phi)\cdot\sin^2(\theta)\cdot\cos(\theta)+b\cdot\sin(\phi)\cdot\sin^3(\theta)-c\cdot\sin^2(\theta) \quad (3)$$

To make it a little easier to handle this equation, the following replacements are done:

$A=-a\cdot\cos^2(\phi)$ $B=-a\cdot\sin(2\cdot\phi)$ $C=-a\cdot\sin^2(\phi)$ $D=c\cdot\cos^2(\phi)$ $E=c\cdot\sin(2\cdot\phi)$
$F=c\cdot\sin^2(\phi)$ $G=-b\cdot\cos(\phi)$ $H=b\cdot\sin(\phi)$

After the replacements, (3) takes the following form:

$$f'(\theta)=A\cdot\cos(\theta)\cdot\sin^2(\theta)+B\cdot\cos^2(\theta)\cdot\sin(\theta)+C\cdot\cos^3(\theta)+\ldots$$
$$..+D\cdot\sin^2(\theta)+E\cdot\sin(\theta)\cdot\cos(\theta)+F\cdot\cos^2(\theta)+G\cdot\sin^2(\theta)\cdot\cos(\theta)+H\cdot\sin^3(\theta)-c\cdot\sin^2(\theta) \quad (4)$$

Using the half angle formulas, cos(ϑ) and and sin(ϑ) can be replaced by:

$$\sin(\vartheta)=\frac{2\cdot\tan\left(\frac{\vartheta}{2}\right)}{1+\tan^2\left(\frac{u}{2}\right)} \qquad \cos(\vartheta)=\frac{1-\tan^2\left(\frac{\vartheta}{2}\right)}{1+\tan^2\left(\frac{\vartheta}{2}\right)}$$ setting $t=\tan\left(\frac{\vartheta}{2}\right)$ (5) and, after doing the calculations and getting rid of denominators the equation f(t) to solve is:

$$f(t)=A\cdot(1-t^2)\cdot 4\cdot t^2+B\cdot(1-t^2)^2\cdot 2\cdot t+C\cdot(1-t^2)^3+D\cdot 4\cdot t^2\cdot(1+t^2)+E\cdot 2\cdot t\cdot(1-t^2)\cdot(1+t^2)+\ldots$$
$$\ldots F\cdot(1-t^2)^2\cdot(1+t^2)+G\cdot 4\cdot t^2\cdot(1-t^2)+H\cdot 8\cdot t^3-c\cdot 4\cdot t^2\cdot(1+t^2)=0$$

or

$$f(t)=4\cdot A\cdot t^2-4\cdot A\cdot t^4+2\cdot B\cdot t-4\cdot B\cdot t^3+2\cdot B\cdot t^5+C-3\cdot C\cdot t^2+3\cdot C\cdot t^4-C\cdot t^6+4\cdot D\cdot t^2+4\cdot D\cdot t^4+..$$
$$..+2\cdot E\cdot t-2\cdot E\cdot t^5+F-2\cdot F\cdot t^2+F\cdot t^4+F\cdot t^2-2\cdot F\cdot t^4+F\cdot t^6+4\cdot G\cdot t^2-4\cdot G\cdot t^4+8\cdot H\cdot t^3-4\cdot c\cdot t^2-4\cdot c\cdot t^4=0$$

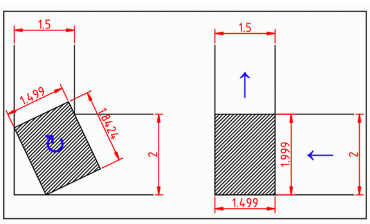
The final equation is:

$$f(t)=(-C+F)\cdot t^6+2\cdot(B-E)\cdot t^5+(-4\cdot A+3\cdot C+4\cdot D-F-4\cdot G-4\cdot c)\cdot t^4+(-4\cdot B+8\cdot H)\cdot t^3+\ldots$$
$$\ldots+(4\cdot A-3\cdot C+4\cdot D-2\cdot F+F+4\cdot G-4\cdot c)\cdot t^2+2\cdot(B+E)\cdot t+C+F=0 \quad (6)$$

NB:Please note that there is a capital C (replacement) and a small c (width of object) in this equation

Equation (6) can be solved for t and then θ can be calculated as: θ=2 * atan(t). If φ is up to 90 degrees, θ_{max} can be up to 90 degrees, which means that atan(t) can be up to 45 degrees, or t_{max} can be up to 1. However when φ is greater than 90°, the value of θ_{max} can be up to 180-φ, so atan(t) can be up to (180-φ)/2 and t_{max} is the tangent of this angle. This fact is useful to filter out unwanted solutions because, at high angles and widths of ladder, equation (6) can have up to 3 positive real roots. The appropriate root is found by a bisection method in the interval (0, t_{max}). Sturm's method can be useful to predict the number of real solutions as well as to confirm that indeed there is a positive root in the interval (0,t_{max}). For comparison, Leguerre 's method gives all the roots of the equation although not always so accurate for the value sought.

After the calculation of the angle θ, the length L is found by plugging θ into (1). Then the lengths X and Y at the critical point, are calculated as: $X=L\cdot\cos(180-\phi-\theta)+\frac{L\cdot\sin(180-\phi-\theta)}{\tan(\phi)}$ $\quad Y=L\cdot\cos(\theta)+X\cdot\cos(\phi)$. X and Y are not coordinates of some sort but lengths along the outer wall.



Note that this problem presupposes that the object will have to go round the corner by touching the corner and the opposite walls. The are instances where the object need not do this motion around the corner but can be moved in straight lines merely requiring change of motion line or it can be rotated not around the edge of the corner but further away. These solutions are not covered by the present method. This can be seen in the adjacent picture where the corridors have widths of 1.5 and 2 whilst the oblong object has a width of 1.499 and will be moved around in a 90 degree corner. If the maximum length is calculated by the present method it will be 1.8424, but, as can be seen in the picture, a length of 1.999 can be accommodated if the object is pushed against the wall and then merely pushed in the other direction without any rotation.

A note on equation (6). In order for the problem to make sense, a and b must be greater than c. Observing that f(-∞)=+INF , f(+∞)=+INF (because of even and positive Leading Coefficient) and $f(0)=F+C=\sin^2(c-a)$ (which is negative since a>c) leads to the conclusion that there are 1,3 or 5 negative real roots and 1,3 or 5 positive real roots for f(t). Sturm's method corroborates this.

REFERENCES

1. http://www.mathematische-basteleien.de/ladder.htm
2. Alasdair McAndrew, "A litany of ladders: easy problems with hard solutions" https://atcm.mathandtech.org/EP2017/invited/4202017_21552.pdf
3. Moretti, Moving a couch around a corner, College Math. J. 33 (2002) 196–200
4. Boute, Raymond T., "Moving a Rectangle around a Corner — Geometricall", https://biblio.ugent.be/publication/314475/file/452143.pdf

Demetrius Papademetriou
demkpap@gmail.com
August 2021