DRAW A COMMON TANGENT TO TWO CIRCLE(S) OR ELLIPSE(S)

Suppose we have two ellipses, the first with semi-axes A_c, B_c, centered at (0,0) and aligned with coordinate axes and the second with semi-axes A_f and B_f, centered at (F_x, F_y) and angle ϑ_f between its axis and x-axis. A tangent line to both ellipses has slope m and intersects the 1st and 2nd ellipses at points (x_c,y_c) and (x_f,y_f) respectively.
Equation of first ellipse with semi-axes A_c and B_c centered at (0,0) and major semi-axis aligned with x-axis:

$$\frac{x^2}{A_c^2}+\frac{y^2}{B_c^2}=1 \quad (1)$$

By differentiating the explicit function (1), the slope of tangent at point (x_c,y_c) is:

$$m=-\frac{B_c^2\cdot x_c}{A_c^2\cdot y_c} \quad (2)$$

The y-intercept L of tangent at point (x_c,y_c) is:

$$L=\pm\sqrt{m^2\cdot A_c^2+B_c^2} \quad \text{(3) (Derivation of this formula at the end of this document)}$$

Equation of the second ellipse with semi-axes A_f and B_f centered a F_x, F_y and angle ϑ between x-axis and major semi-axis:

$$A\cdot x^2+B\cdot x\cdot y+C\cdot y^2+D\cdot x+E\cdot y+F=0 \quad \text{(4) where:}$$

$$A=A_f^2\cdot\sin^2(\vartheta)+B_f^2\cdot\cos^2(\vartheta)$$
$$B=2\cdot(B_f^2-A_f^2)\cdot\sin(\vartheta)\cdot\cos(\vartheta)$$
$$C=A_f^2\cdot\cos^2(\vartheta)+B_f{\cdot}^2\sin^2(\vartheta)$$
$$D=-2\cdot A\cdot F_x-B\cdot F_y$$
$$E=-B\cdot F_x-2\cdot C\cdot F_y$$
$$F=A\cdot F_x^2+B\cdot F_x\cdot F_y+C\cdot F_y^2-A_f^2\cdot B_f^2$$

By taking the first derivative of implicit equation (4), we find that the slope m of tangent at point x_f,y_f is :

$$m=-\frac{(2\cdot A\cdot x_f+B\cdot y_f+D)}{(B\cdot x_f+2\cdot C\cdot y_f+E)} \quad (5)$$

The equation of the tangent line is:

$$y=m\cdot x+L \quad \text{so} \quad y_f=m\cdot x_f+L \quad \text{(5a).}$$

Plugging (5a) into (5) we get:

$$A\cdot x_f^2+B\cdot x_f\cdot(m\cdot x_f+L)+C\cdot(m\cdot x_f+L)^2+D\cdot x_f+E\cdot(m\cdot x_f+L)+F=0 \quad (6)$$

Rearranging (6) to solve for x_f we get:

$$x_f^2\cdot(A+B\cdot m+C\cdot m^2)+x_f\cdot(B\cdot L+2\cdot m\cdot C\cdot L+D+E\cdot m)+C\cdot L^2+E\cdot L+F=0 \quad (7)$$

In order for the line to be tangent to the ellipse, quadratic equation (7) must have one (1) real double solution which means that its discriminant must be zero, so:

$$(B\cdot L+2\cdot m\cdot C\cdot L+D+E\cdot m)^2-4(A+B\cdot m+C\cdot m^2)\cdot(C\cdot L^2+E\cdot L+F)=0 \quad (8)$$

After expanding (8) we replace L with (3), doing some squaring to get rid of the radical, we arrive at the following quartic equation which can be solved for m:

$$m^4\cdot(N^2\cdot A_c^4+T^2+2\cdot N\cdot T\cdot A_c^2-Q^2\cdot A_c^2)+m^3\cdot(2\cdot N\cdot W\cdot A_c^2+2\cdot T\cdot W-2\cdot P\cdot Q\cdot A_c^2) \quad +$$
$$m^2\cdot(2\cdot N^2\cdot A_c^2\cdot B_c^2+W^2+2\cdot N\cdot S\cdot A_c^2+2\cdot N\cdot T\cdot B_c^2+2\cdot S\cdot T-A_c^2\cdot P^2-Q^2 B_c^2) \quad +$$
$$m\cdot(2\cdot N\cdot W\cdot B_c^2+2\cdot S\cdot W-2\cdot P\cdot Q\cdot B_c^2)+(N^2\cdot B_c^4+2\cdot N\cdot S\cdot B_c^2-B_c^2\cdot P^2+S^2)=0 \qquad (9)$$

where:

$$N=B^2-4\cdot A\cdot C$$
$$P=2\cdot B\cdot D-4\cdot A\cdot E$$
$$Q=4\cdot C\cdot D-2\cdot B\cdot E$$
$$S=D^2-4\cdot A\cdot F$$
$$T=E^2-4\cdot C\cdot F$$
$$W=2\cdot D\cdot E-4\cdot B\cdot F$$

Solving (9) produces 0, 1, 2, 3 or 4 real solutions for m.
After calculating the slope m, we must find the tangent points in both ellipses. For this purpose we will use the parametric form of the ellipses.
For the first ellipse:

$$x_c = A_c \cdot \cos(t_c) \quad (10)$$
$$y_c = B_c \cdot \sin(t_c) \quad (11)$$

and for the second:

$$x_f = A_f \cdot \cos(\theta_f) \cdot \cos(t_f) - B_f \cdot \sin(\theta_f) \cdot \sin(t_f) + F_x \quad (12)$$
$$y_f = A_f \cdot \sin(\theta_f) \cdot \cos(t_f) + B_f \cdot \cos(\theta_f) \cdot \sin(t_f) + F_y \quad (13)$$

where t_c and t_f are the parameters for the first and second ellipse respectively.
The slope of tangent at any point of a function in parametric form is given by the ratio of first derivatives of y and x in respect to parameter t. So for the 1st ellipse we have:

$$m = \frac{(B_c \cdot \sin(t_c))'}{(A_c \cdot \cos(t_c))'} \quad \text{or} \quad m = -\frac{B_c \cdot \cos(t_c)}{A_c \cdot \sin(t_c)} \quad (14)$$

And for the 2nd ellipse:

$$m = \frac{(A_f \cdot \sin(\theta_f) \cdot \cos(t_f) + B_f \cdot \cos(\theta_f) \cdot \sin(t_f) + F_y)'}{(A_f \cdot \cos(\theta_f) \cdot \cos(t_f) - B_f \cdot \sin(\theta_f) \cdot \sin(t_f) + F_x)'} \quad \text{or} \quad m = \frac{A_f \cdot \sin(\theta_f) \cdot \sin(t_f) - B_f \cdot \cos(\theta_f) \cdot \cos(t_f)}{A_f \cdot \cos(\theta_f) \cdot \sin(t_f) + B_f \cdot \sin(\theta_f) \cdot \cos(t_f)} \quad (15)$$

we will calculate t_c and t_f using (14) and (15) and then calculate x_c,y_c and x_f,y_f using (10),(11) and (12),(13) respectively.

Using the trigonometric identities for half-angle: $\sin(\phi) = \frac{2 \cdot \tan(\phi/2)}{1 + \tan^2(\phi/2)}$ and $\cos(\phi) = \frac{1 - \tan^2(\phi/2)}{1 + \tan^2(\phi/2)}$

and by putting $w = \tan(t_c/2)$, equation 14 becomes:

$$m = \frac{-B_c \cdot (1 - w^2)}{2 \cdot A_c \cdot w} \quad (16)$$

Re-arranging (16) and solving for w we get:

$$B_c \cdot w^2 - 2 \cdot A_c m \cdot w - Bc = 0 \quad (17)$$

Using the same trigonometric identities as above and putting $u = \tan(t_f/2)$ equation (15) becomes:

$$m = \frac{B_f \cdot \cos(\theta_f) \cdot u^2 + 2 \cdot A_f \cdot \sin(\theta_f) \cdot u - B_f \cdot \cos(\theta_f)}{-B_f \cdot \sin(\theta_f) \cdot u^2 + 2 \cdot A_f \cdot \cos(\theta_f) \cdot u + B_f \cdot \sin(\theta_f)} \quad (18)$$

Re-arranging (18) and solving for u we get:

$$B_f \cdot [\cos(\theta_f) + m \cdot \sin(\theta_f)] \cdot u^2 + 2 \cdot A_f \cdot [\sin(\theta_f) - m \cdot \cos(\theta_f)] \cdot u - B_f \cdot [\cos(\theta_f) + m \cdot \sin(\theta_f)] = 0 \quad (19)$$

(17) and (19) can be solved producing 2 solutions for w and u from which the t_c and t_f are calculated:

$$t_c = 2 \cdot \operatorname{atan}(w)$$
$$t_f = 2 \cdot \operatorname{atan}(u)$$

The actual coordinates for each ellipse are calculated from (10)-(11) or (12)-(13).
Out of the four possible combinations the correct line (or lines since there might exist two lines depending on the symmetry of ellipses) is selected if the coordinates satisfy the following equation:

$$m = \frac{y_f - y_c}{x_f - x_c} \quad (20)$$

The procedure to find the endpoints of the tangent lines is repeated for each real value of slope m as calculated from (9).

The above described procedure assumes that the first ellipses is centered at (0,0) with with major axis parallel to x-axis.

If this is not the case, both ellipses are first translated so that the first ellipse is centered at (0,0) and then rotated around (0,0) so that the major axis of the first ellipse is aligned with x-axis. After the calculations finish, the two figures withs the tengents are rotated and translated back to their original coordinate.

This program does not aspire to present a bullet proof solution for common tangent to ellipses. The user might discover that no solutions are found where clearly there are some, due to round off or other types of errors. For this I must apologize. Please use with caution. The calculation error increases as the two curves are several orders of magnitude apart relative to their size.

For the solution of the quartic equation, the NBS method is applied together with a trigonometric method for the cubic equation (both compiled after the instructions given by David Wolters in www.quarticequations.com). T

For the presentation of solution, the DXFighter program in php was used (https://github.com/enjoping).

For any comments or errors or a copy of the php code, please drop an e-mail.

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This is an example of calculations for two ellipses in case someone wants to type in any of the above formulas. The 1st ellipse is placed at (0,0) so there is no need for rotation or translation. Also there is no root refinement at all.

DATA INPUT:

First ellipse:

Ac=3.1
Bc=1.9
Cx=0
Cy=0
Ca=0

Second ellipse:

Af=4.7
Bf=2.4
Fx=11.8
Fy=5.9
Fa=57.6

CALCULATIONS:

Parameters of 2nd ellipse:

A=17.401487915628820247
B=-14.775825766770100955
C=10.448512084371181174
D=-323.49774278489655899
E=51.062301452307266914
F=1630.7644931465833906

Intermediate variables:

N=-508.95359999999999445
P=6005.6524799999988318
Q=-12011.304959999997664
S=-8860.12489598421962
T=-65548.891424015790108
W=63346.489547029595997

Coefficients of quartic equation:

coef4=3575335911.9112033844 (coefficient of m^4)
coef3=-7537796674.0714588165
coef2=4652400672.8339385986
coef1=-834471555.6923995018
coef0=-15769600.070743814111

After dividing by coef4, the coefficients for the quartic equation are:

1,-2.1082764975898764526,1.3012485504745170406,-0.23339668670358051927,-0.0044106625109566676909

Cubic equation coefficients:

1,-0.36556262088417157408,0.030868210562315492873,-0.00013667906580878177438

Cubic equation 3 real solutions:

0.23858165621900701003, 0.12229660142414131019,0.0046843632410233093655

We choose 0.23858165621900701003 as the largest positive root for finding the roots of the quartic, which are:

Quartic equation 4 real solutions

1.1681320830908858976, 0.57677625222923922799,0.38056981463943717614,-0.017201652369685738186

After finding the points of tangency on both ellipses we end up with following 4 lines:

Line from 10.530465255092712695,8.2115785233194120707 to 2.7450875122265441242,-0.882770997104268762
Line from 12.187330996120490312,9.6383795438182833237 to -2.1244863327229581706,1.3836631822984839513
Line from 10.864248090142947234,1.8981235341962767738 to 1.6352812074651390528,-1.6141426816579269587
Line from 9.9991366138945210196,1.7287464882153857904 to 0.086969900973839386893,1.8992521342414185881
(Note: the first point on each line is on the 2nd ellipse)

PROOF OF THE EQUATION (3): $\mathbf{L=\pm\sqrt{m^2\cdot A_c^2+B_c^2}}$

where L is the y-intercept of the tangent line with slope m to ellipse centered at (0,0) with semi-axes Ac, Bc. x_c,y_c are the coordinates of the point of contact on the ellipse (Courtecy Mr. David Wolters).

This expression for L is derived by applying (1) and (2) to L = yc- m·x_c as follows.

$$L=y_c\left(1-\frac{m\cdot x_c}{y_c}\right)=y_c\left(1+\frac{B_c^2\cdot x_c^2}{A_c^2\cdot y_c^2}\right)\ (3.1)$$

$$L=y_c\left[1+\frac{B_c^2}{y_c^2}\left(1-\frac{y_c^2}{B_c^2}\right)\right]=\frac{B_c^2}{y_c}\ (3.2)$$

The product of (3.1) and (3.2), together with the square of equation (2), gives the square of equation (3):

$$L^2=B_c^2\left(1+\frac{B_c^2\cdot x_c^2}{A_c^2\cdot y_c^2}\right)=\left(B_c^2+A_c^2\frac{B_c^4\cdot x_c^2}{A_c^4\cdot y_c^2}\right)=B_c^2+A_c^2\cdot m^2=m^2\cdot A_c^2+B_c^2.$$

After we find values of m and L that both satisfy (3) and produce a line that is tangent to the second ellipse, then we can find the tangent point (x_c, y_c) ellipse. The component y_c is given by (3.2):

$$y_c=\frac{B_c^2}{L}.$$

This result, together with L = y_c- m·x_c and equation (3), produce the corresponding equation for x_c.

$$x_c=\frac{y_c-L}{m}=\frac{\frac{B_c^2}{L}-L}{m}=\frac{B_c^2-L^2}{m\cdot L}=\frac{B_c^2-m^2\cdot A_c^2-B_c^2}{m\cdot L}=\frac{-A_c^2\cdot m}{L}$$